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TOPOLOGICAL PROPERTIES OF G-HAUSDORFF METRIC

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ABSTRACT

Consider a $G-$ metric space $(\mathcal{M}, \Upsilon)$, we define a general metric space with general Hausdorff metric Γ on the set Ҡ *of the class of all non-empty compact subsets of $\mathcal{M}$.We show that if $(\mathcal{M}, \Upsilon)$is complete, then the Hausdorff metric space* (Ҡ, Γ) *is also complete $G-$ metric space. Similarly, the compactness. Illustrative example by using Mat lab is presented for a Cuachy sequence in* (Ҡ, Γ) *also converges to the element in*(Ҡ, Γ).

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INTRODUCTION AND PRELIMINARIES1

The Hausdorff metric is defined on the space of nonempty closed bounded subsets of a metric space by F. Hausdorff, see[1].One can review the study presented recently by Albundi and Abd [2] to see important aspects of employing this distance in fractals and its applications. Here, we prove some properties of generalized Hausdorff metric under G-metric space.

Definition 1.1 [3]

Let $\mathcal{M}$ be a nonempty set and:$\mathcal{M}\times \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function satisfying the following condition

- $\Upsilon(r, s, t) = 0$ if and only if $r = s = t$,
- $0 < \Upsilon(r, r, s), \forall\, r, s \in \mathcal{M}$ with $r \neq s$,
- $\Upsilon(r, r, s) \leq \Upsilon(r, s, t)$ for all $r, s, t \in \mathcal{M}$ with $s \neq t$,
- $\Upsilon(r, s, t) \leq \Upsilon(r, t, s) = \ldots$, (symmetry in all three vairables),
- $\Upsilon(r, s, t) \leq \Upsilon(r, a, a) + \Upsilon(a, s, t)$ for all $r, s, t, a \in \mathcal{M}$.

Then the function Υ is called generalized metric on $\mathcal{M}$, and the pair $(\mathcal{M}, \Upsilon)$ is called a G-metric space.

Definition 1.2[4]

Let $(\mathcal{M}, \Upsilon)$ be a Υ -metric space, then the G –metric is called a symmetric if

$\Upsilon(r,s,s) = \Upsilon(r,r,s)$ for all $r,s,t \in \mathcal{M}$

Many results and examples about ϒ-metric space and its generalization one can found in [4-14]

Definition 1.3[3]

Let $(\mathcal{M},\Upsilon)$ be a ϒ-metric space then for $r_0 \in \mathcal{M}, \epsilon > 0$ the ϒ-ball with center r_0 and radiusϵ is $B_\Upsilon(r_0, r_0, \epsilon)$ where

$B_\Upsilon(r_0,r_0,\epsilon) = \{ s \in \mathcal{M} : \Upsilon(r_0,s,s) < \epsilon \}$.

Definition 1.4 [3]

Let $(\mathcal{M},\Upsilon)$ be a $G-$metric space and $\{r_n\}$ be a sequence of points in $\mathcal{M}$, ifthere exists $k \in N, \epsilon > 0$ for $m,n,l \geq k$ then the sequence $\{r_n\}$ is said to be

- $\Upsilon-$convergent to μ if $\Upsilon(r, r_n, r_m) < \epsilon$ for all $m,n \geq k$.
- 2-ϒ -Cauchy if$\Upsilon(r_n,r_m,r_l) < \epsilon$ for all $\forall$ $m,n,l \geq k$.

Lemma 1.5[12]

If $\{r_n\}$ be a bounded sequence in G-metric space with $\Upsilon-$bound $\mathcal{M}$ and for each n,m in $N, m > n$, and $0 \leq \lambda < 1$ such that

$\Upsilon(r_n, r_{n+1}, r_m) \leq \lambda^n M$

Definition 1.6 [3]

A G–metric space $(\mathcal{M},\Upsilon)$ is complete if every ϒ -Cauchy sequence $(\mathcal{M},\Upsilon)$ is ϒ - convergent in $(\mathcal{M},\Upsilon)$.

Remark 1.7[10]: Every G–metric $(\mathcal{M},\Upsilon)$ on $\mathcal{M}$ defines a metric d_Υ on $\mathcal{M}$ [8] given by

$d_\Upsilon(r,s) = \Upsilon(r,s,s) + \Upsilon(s,r,r)$ for all $r,s \in \mathcal{M}$.

Proposition 1.8 [10]

A G-metric space $(\mathcal{M},\Upsilon)$ is ϒ -complete if and only if $(\mathcal{M},d_\Upsilon)$ is a complete metric space.

Proposition 1.9 [13]

Let $(\mathcal{M},\Upsilon)$ be a G-metric space the following statements are equivalent

- $\{r_n\}$ is ϒ-convergent to r, if and only if$\Upsilon(r_n,r_n,r) \rightarrow 0$ as $n \rightarrow \infty$,
- is$\Upsilon(r_n, r, r) \rightarrow 0$ as $n \rightarrow \infty$ if and only if$\Upsilon(r_n,r_m,r) \rightarrow 0$ as $m,n \rightarrow \infty$.

Proposition 1.10[3]: Let $(\mathcal{M},\Upsilon)$ be a ϒ-metric space, then the following are equivalent :

- $(\mathcal{M},\Upsilon)$ is symmetric.
- $\Upsilon(r,s,s) \leq \Upsilon(r,s,a)$ for all $r,s,a \in \mathcal{M}$,
- $\Upsilon(r,s,t) \leq \Upsilon(r,s,a) + \Upsilon(t,s,b)$ for all $r,s,t\ a,b \in \mathcal{M}$.

Proposition 1.11 [7]

let $\{r_n\}$ and $\{s_n\}$ be a sequence in a G-metric space ($\mathcal{M}$,ϒ) if $\{r_n\}$ converges to r and $\{s_n\}$ converge to s then $\Upsilon(r_n, r_n, s_n)$ converges to $\Upsilon(r, r, s)$.

Definition 1.12 [14]

The self- map f on a ϒ -metric space ($\mathcal{M}$, ϒ) is ϒ - continuous at $r \in \mathcal{M}$

If and only if every sequence

$\{r_n\}_{n=1}^{\infty} \subset \mathcal{M}$, with$r_{n\rightarrow}r$, we have $f_{r_n} \overset{\Upsilon}{\rightarrow} f_r$

Now present the following important classes of subset of ϒ -metric space$\mathcal{M}$.

$2^{\mathcal{M}}$ =The family of all nonempty subset of$\mathcal{M}$.

$CB(\mathcal{M})$ = The family of all nonempty closed and bounded subsets of $\mathcal{M}$.

$K(\mathcal{M})$ = The family of all nonempty compact subsets of $\mathcal{M}$.

We now define the general Hausdorff ϒ- metric on $CB(\mathcal{M})$

Definition 1.13 [11]: For $a \in \mathcal{M}$ and $R, S, T \in K(\mathcal{M})$is define:

$\mu(a, S, T) = \inf \{\Upsilon(a, s, t) : s \in S, t \in T\}$

and $\rho(R, S, T) = \sup\{\mu(a, S, T) : a \in R\}$

the ϒ −Hausdroff distance on $K(\mathcal{M})$among R, S, T is

$\Gamma(R, S, T) = \max\{\sup_{a\in R}\Upsilon(a, S, T), \sup_{a\in T}\Upsilon(a, R, S), \sup_{a\in S}\Upsilon(a, R, T)\}$

where,

$\Upsilon(a, S, T) = d_\Upsilon(a, S) + d_\Upsilon(S, T) + d_\Upsilon(a, T)$,

$d_\Upsilon(a, S) = \inf\{d_\Upsilon(a, s): s \in S\}$,

and $d_\Upsilon(r, s) = \inf\{d_\Upsilon(a, s): a \in R, s \in S\}$.

We explain the ϒ −Hausdroff distance on $K(\mathcal{M})$ by the following example.

Example 1.14

Consider the complete G-metric space ($\mathcal{M}$, ϒ) [4], where $\mathcal{M} = \mathbb{R}$, and

$\Upsilon(r, s, t) = |r - s| + |s - t| + |t - r|$, for all $r, s, t \in \mathcal{M}$,

Let $R = [0, 5], S = [7, 10], T = [12, 20]$

We calculate the Hausdorff distance $\Gamma(R, S, T)$ as follows

Since $d_\Upsilon(r, s) = \Upsilon(r, s, s) + \Upsilon(s, r, r)$.

Then$d_\Upsilon(s, t) = \Upsilon(s, t, t) + \Upsilon(t, t, s), s \in S, t \in T$,

$=2|s-t|+2|t-s|=4|s-t|$

$d_\Upsilon(S,T)=\inf\{4|s-t| : s\in S, t\in T\}$, then$4|12-10|=8$

$d_\Upsilon(r,S)=\inf\{4|r-s| : s\in S\}$, then $4|r-7|=4|0-7|=28$

$d_\Upsilon(r,T)=\inf\{4|r-t| : t\in T\}$, then $4|r-12|=4|0-12|=48$

$\sup_{r\in R}\Upsilon(r,S,T)=\sup\{8+4|r-7|+4|r-12|\}=84$

Similarly, $\sup_{s\in S}\Upsilon(s,R,T)=56$ and $\sup_{t\in T}\Upsilon(t,R,S)=44$

So, $\Gamma(R,S,T)=\max\{\sup_{r\in R}\Upsilon(r,S,T), \sup_{t\in T}\Upsilon(t,R,S), \sup_{s\in S}\Upsilon(s,R,T)\}=84$.

2. MAIN RESULTS

In the following, we give some properties of Γ

Proposition 2.1

If R, S and T are nonempty subsets of $\Upsilon-$ metric space $(\mathcal{M},\Upsilon)$ then

$\Gamma(R,S,T)=\inf\{\epsilon>0\ R\subseteq N_\epsilon(S\cup T), S\subseteq N_\epsilon(R\cup T), T\subseteq N_\epsilon(R\cup S)$

Proof

If

$\{\epsilon>0\ R\subseteq N_\epsilon(S\cup T), S\subseteq N_\epsilon(R\cup T), T\subseteq N_\epsilon(R\cup S)\}\neq\emptyset$

Suppose that

$\varphi=\{\epsilon>0\ R\subseteq N_\epsilon(S\cup T), S\subseteq N_\epsilon(R\cup T), T\subseteq N_\epsilon(R\cup S)\}<\infty$

Then ϵ satisfies

$R\subseteq N_\epsilon(S\cup T)$, $S\subseteq N_\epsilon(R\cup T)$ and $T\subseteq N_\epsilon(S\cup R)$, for all $r\in R$ we have

$\Upsilon(r,S,T)<\epsilon$

Take sup we get

$\sup_{r\in R}\Upsilon(r,S,T)<\epsilon$ so $\Upsilon(s,R,T)<\epsilon$ and $\sup_{s\in S}\Upsilon(s,R,T)<\epsilon$

And $\Upsilon(t,R,S)<\epsilon$ then $\sup_{t\in T}\Upsilon(t,R,S)<\epsilon$

So

$\Gamma(R,S,T)=\max\{\sup_{r\in R}\Upsilon(r,S,T), \sup_{s\in S}\Upsilon(s,R,T), \sup_{t\in T}\Upsilon(t,R,S)\}<\epsilon$By definition

Thus $\Gamma(R,S,T)\leq\varphi$

On the other hand

If $\Gamma(R,S,T)<\epsilon$, Then $R\subseteq N_\epsilon(S\cup T)+\epsilon, S\subseteq N_\epsilon(R\cup T)+\epsilon, T\subseteq N_\epsilon(R\cup S)+\epsilon$. So

$\Gamma(R,S,T)=\inf\{\epsilon>0, R\subseteq N_\epsilon(S\cup T), S\subseteq N_\epsilon(R\cup T), T\subseteq N_\epsilon(R\cup S)\}$.

Proposition 2.2

Let$(\mathcal{M}, \Upsilon)$ be a Υ-metric space then for any nonempty subsets R, S and T

of $\mathcal{M}$,

$$\Gamma(R, S, T) = \sup_{\substack{a\in R\\ a\in S\\ a\in T}} \{|d_\Upsilon(a, R) - d_\Upsilon(a, S)| + |d_\Upsilon(a, S) - d_\Upsilon(a, T)| + |d_\Upsilon(a, T) - d_\Upsilon(a, R)|\}$$

Proof:

since $d_\Upsilon(a, S) \leq d_\Upsilon(a, s)$for all $s \in S$, Then

$d_\Upsilon(a, S) - d_\Upsilon(a, r) \leq d_\Upsilon(a, s) - d_\Upsilon(a, r) \leq d_\Upsilon(r, s)$

$= d_\Upsilon(R, S)$ for all $r \in R, s \in S$

$= \Upsilon(a, R, S) \leq \Gamma(R, S, T)$ ... (2.1)

and

$d_\Upsilon(a, R) \leq d_\Upsilon(a, r)$for all $r \in R$

$d_\Upsilon(a, R) - d_\Upsilon(a, T) \leq d_\Upsilon(a, r) - d_\Upsilon(a, t) = d_\Upsilon(a, T) = d_\Upsilon(R, T)$ for all $r \in R, t \in T$

$\Upsilon(a, R, T) \leq \Gamma(R, S, T)$ ... (2.2)

also

$d_\Upsilon(a, T) \leq d_\Upsilon(a, t)$ for all $t \in T$

$d_\Upsilon(a, T) - d_\Upsilon(a, s) \leq d_\Upsilon(a, t) - d_\Upsilon(a, s) \leq d_\Upsilon(t, s)$ for all $s \in S, t \in T$

$d_\Upsilon(T, S) \leq \Upsilon((a, S, T) \leq \Gamma(R, S, T)$…(2.3)

From(2.1), (2.2) and (2.3) we have

$|d_\Upsilon(a, S) - d_\Upsilon(a, R)| + |d_\Upsilon(a, S) - d_\Upsilon(a, T)| + |d_\Upsilon(a, T) - d_\Upsilon(a, R)| \leq \Upsilon(a, R, S) + \Upsilon(a, S, T) + \Upsilon(a, R, T)$

Take sup to both side we have

$\sup\{|d_\Upsilon(a, R) - d_\Upsilon(a, S)| + |d_\Upsilon(a, S) - d_\Upsilon(a, T)| + |d_\Upsilon(a, t) - d_\Upsilon(a, R)| \leq$

$\sup_{a\in R}\Upsilon(a, S, T)$,$\sup_{a\in S}\Upsilon(a, R, S), \sup_{a\in T}\Upsilon(a, R, T), \leq \Gamma(R, S, T)$.

Proposition 2.3

Let $(\mathcal{M}, \Upsilon)$ be a G- metric space and let R be a closed subset of $\mathcal{M}$. if $\{r_n\}$ Υ-converges to r and $r_n \in R\ \forall\ n$ Then $r \in R$.

Proof. Suppose $\{r_n\}$ is a sequence that$\Upsilon -$ converges to rand $r_n \in R$ for all n. There are two cases to consider. If there exists a positive integer n such that $r_n = r$, then it is clear $r \in R$

If there does not exist a positive integer n such that $r_n = r$ then r is a limit point of R Since R is closed $r \in R$.

Proposition 2.4: Let $x \in \mathcal{M}$and let $R, S, T \in K(\mathcal{M})$

- $\mu(x,R,R)=0$ if and only if $x\in R$.
- $\rho(R,S,T)=0$ if and only if $R\subseteq S\subseteq T$.
- There exists$r_x\in R$ such that $\mu(x,R,R\ )=\Upsilon(x,r_x,r_x)$.
- iv-There exists$r^*\in R, s^*\in S\ and\ t^*\in T$ such that $\rho(R,S,T)=\Upsilon(r^*,s^*,t^*)$.
- If $R\subseteq S\subseteq T$ then $\mu(x,T,T)\leq\mu(x,S,S)\leq\mu(x,R,R)$.
- If $S\subseteq T\subseteq Z$ then $\rho(R,Z,Z)\leq\rho(R,T,T)\leq\rho(R,S,S)$.
- $\rho(R\cup S\cup T,Z)=max\ \{\Upsilon(R,Z,Z),\Upsilon(S,Z,Z),\Upsilon(T,Z,Z)\}$.
- $\rho(R,S,T)\leq\Upsilon(a,Z,Z)+\Upsilon(Z,S,T)$.

Proof

I- Suppose $x\in R$, then the Infimum Distance is $\Upsilon\ (x,r,r)\ =\ 0$, Where $x\ =r$, now, suppose that $\mu(x,R,R)=0$, then for each positive integer n, there exists$a_n\ \in R$

such that

$\Upsilon\ (x, a_n ,a_n\ )<\frac{1}{n}$ (By definition $\{r_n\}$ converges to r).

Since R compact, it is closed (by proposition (2.3))

It is follow that $x\in\ R$.

II. Suppose that $R\subseteq S\subseteq T$, let $a\in R$, Since $a\in S, a\in T$ (part i), we have

$\mu(a,S,T\ )\ =\ 0,$

therefore,

$\rho(\ R,S,T\ )=\mathrm{su\,p}\{\mu(a,S,T): a\in R\}=0.$

To prove the converse, suppose $\rho(R,S,T)=0$. let $a\in R$, then $0\leq\mu(a,S,T)\leq\rho(R,S,T)=0$

and thus by property (i), we obtain, $a\in S, a\in T$

it follows that $R\subseteq S\subseteq T$.

III. By Definition of an Infimum, Let a Sequence $\{a_n\}$ in R such that

$\Upsilon(x,a_n,a_n)\leq\mu(x,R,R)+\frac{1}{n}$

We know R is a compact. so, there exist a subsequence $\{a_{nk}\}$ of $\{a_n\}$ that converges to anelement $a_x\in R$,we get

$\mu(x,R,R)\leq\Upsilon\big(x,a_{x,}a_x\big)$

$\leq\Upsilon\big(x,a_{nk,}a_{nk}\big)+\Upsilon\big(a_{nk,},a_{x,}a_x\big)$

$\leq\mu(x,R,R)+\frac{1}{n_k}+\Upsilon\big(a_{nk,},a_{nk,}a_x\big)$

Since

$\mathrm{Lim}_{n\to\infty}\frac{1}{n_k}+\Upsilon\left(a_{nk,},a_{nk,}a_x\right)=0.$

It follows that,

$\Upsilon\left(x,a_{x,}a_x\right)=\mu(x,R,R)$

IV. By Definition of Supremum, let{ r_n} be a Sequence in R. Such that

$\rho(R,S,T)=\lim_{n\to\infty}\mu(r_n,S,T)$ By property (iii)

there exists a sequence $\{s_n\}$in S and $\{t_n\}$in T such that, $\mu(r_n,S,T)=\Upsilon(r_n,s_n,t_n)$

Since R,S a compact, there exists a subsequences $\{r_{nk}\}$ of $\{r_n\}$ that converges to element $r^*\in R$, since Ssubsequentially

compact, there exist a sub sequentially $\{s_{nkj}\}$ of $\{s_{nk}\}$ thatconverges to s^*, similarly $\{t_{nkj}\}$ of $\{t_{nk}\}$ converges tot^*.

(Proposition 1.11)

We know that $\Upsilon\left(r_{nkj},s_{nkj},t_{nkj}\right)$ converges to $\Upsilon(r^*,\ s^*,t^*)$.

Therefore, it follows that

$\rho(R,S,T)=\lim_{j\to\infty}\mu\left(r_{nkj},S,T\right)=\lim_{j\to\infty}\mu(r_{nkj},s_{nkj},t_{nkj})=\Upsilon(r^*,\ s^*,t^*)$

V. Suppose that $R\subseteq S\subseteq T$and $x\in\mathcal{M}\in\mathcal{M},t\ a\in R$ then $a\in S,a\in T$

It follows that

$\Upsilon(x,a,a)\geq\{\inf\{\Upsilon(x,s,s):\ s\in S\}=\ \mu(x,S,S)$

since this is true for all $a\in R$,we have

$\mu(x,R,R)=\inf\{\Upsilon(x,a,a):a\in R\}\geq\mu(x,S,S)$

similarly,

$\mu(x,S,S)\ =\inf\{\Upsilon(x,s,s):\ s\ \in S\}\geq\mu(x,T,T)$

then

$\mu(x,T,T)\ \leq\ \mu(x,S,S)\ \leq\ \mu(x,R,R).$

VI- Suppose that $S\subseteq T\subseteq Z$

By part (v), implies that

$\mu(a,Z,Z)\leq\mu(a,T,T)\leq\mu(a,S,S)\ \forall a\in R$

it follows that

$\sup\{\mu(a,Z,Z):a\in R\}\leq\sup\{a,T,T):a\ \in R\}\leq\sup\{\mu(a,S,S):a\ \in R\}$

Thus

$\Upsilon(R, Z, Z) \leq \Upsilon(R, T, T) \leq \Upsilon(R, T, T)$.

VII- By Definition of r and ρ we see that

$\Upsilon(R \cup S \cup T, Z) = \sup\{\mu(x, x, z) : x \in R \cup S \cup T\}$

$= \max\{\sup(\mu(x, x, Z) : x \in R), \sup(\mu(x, x, Z) : x \in S))$

$, \sup(\mu(x, x, Z) : x \in T)\}$

$= \max\{\Upsilon(R, Z, Z), \Upsilon(S, Z, Z), \Upsilon(T, Z, Z)\}$

VIII- From part (iii) for all $a \in R$, there exists $Z_a \in Z$ such that $\mu(a, Z, Z) = \Upsilon(a, Z_a, Z_a)$,

then, we have

$\mu(a, S, T) = \inf\{\Upsilon(a, s, t) : s \in S, t \in T\}$

$\leq \inf\{\Upsilon(a, Z_{a,} Z_{a,}) + \Upsilon(Z_a, s, t)\}$

$= \mu(a, Z_a, Z_a) + \mu(Z_a, S, T)$

since $a \in R$ (a is arbitrary),take supremum, we get

$\sup(R, S, T) \leq \Upsilon(a, Z, Z) + \Upsilon(Z, S, T)$.

Completeness of K($\mathcal{M}$)

Firstly we must prove that (K(□),Γ) is G-metric space.

Theorem 2.5

Let$(\mathcal{M}, \Upsilon)$ be a G–metric space then ((K□),Γ) is also ϒ- metric space.

Proof

To satisfy the condition of definition (1.1) Clear condition (1) holds,

Since$\sup_{a\in R}\Upsilon(a, S, T)$, $\sup_{a\in T}\Upsilon(a, R, S)$, $\sup_{a\in S}\Upsilon(a, R, T)$nonnegative, it follows that

$\Gamma(R\, S, T) \geq 0$ for all $R, S, T \in \mathcal{M}$

For second property, suppose $R = S = T$,therefore $R \subseteq S \subseteq T$ and $T \subseteq S \subseteq R$

By peoperty (ii) proposition2.4 we find

$\Upsilon(r, s, t) = 0$ and $\Upsilon(t, s, r) = 0$

Thus $\Gamma(R, S, T) = 0$

Now, suppose that $\Gamma(R, S, T) = 0$

This implies

$\Upsilon(r, s, t) = \Upsilon(t, s, r) = 0$

Then $R \subseteq S \subseteq T$ and $T \subseteq S \subseteq R$, by property (ii) proposition2.4

Hence $R = S = T$

The third property can be proved from symmetry of the definition.

Since

$\Gamma(R, S, T) = \max\{\sup_{x\in R}\Upsilon(x, S.T), \sup_{x\in T}\Upsilon(x, R, S), \sup_{x\in S}\Upsilon(x R, T)\}$

$= \max\{\sup_{x\in R}\Upsilon(x, T, S), \sup_{x\in T}\Upsilon(x, S, R), \sup_{x\in S}\Upsilon(x T, R)\} = \Gamma(T, S, R)$.

To prove (4) :by property (iii) proposition(2.4),guarantees that for each $a \in R$,

there exists $Z_a \in Z$ such that

$\mu(a, Z, Z) = \Upsilon(a, Z_a Z_a)$

$\mu(a, S, T) = \inf\{\Upsilon(a, s, t) : s \in S, t \in T\} \leq \inf\{\Upsilon(a, Z_a, Z_a) + \Upsilon(Z_a, s, t)\}$

$\mu(a, Z_a, Z_a) + \mu(Z_{a,} S, T) \;\forall\, a \in R$,

Taking supremum we find

$\sup(R, S, T) \leq \Upsilon(a, Z, Z) + \Upsilon(Z, S, T)$.

Hence $(\mathcal{M}, \Gamma)$ is Υ- metric space.

To verify the completeness of K(□) we need to prove some proposition as preliminary steps:

Definition2.6

For $\epsilon > 0$ we define $R + \epsilon$ as the following

$R + \epsilon = \{x \in \mathcal{M}: \Upsilon(x, R, z) < \epsilon\}$ for fix z in $\mathcal{M}$.

Proposition 2.7

$R + \epsilon$ is closed for all possible choices of $R \in \mathcal{M}$ and $\epsilon > 0$.

Proof

Let $R \in \mathcal{M}$ and $\epsilon > 0$. additionally, let r be a limit point of $R + \epsilon$,

Then there existsa sequence $\{r_n\}$ of points in $(R + \epsilon)\backslash\{r\}$ that converges to r.

Since $r_n \in R + \epsilon \;\forall n$, by definition

$\mu(r_n, R, R) \leq \epsilon$ for all n.

Also for each n there existsa$_n \in R$, (Bypart (ii) proposition(2.4))

such that

$\mu(r_n, R, R) = \Upsilon(r_n, a_n, a_n)$. Thus $\Upsilon(r_n, a_n, a_n) \leq \epsilon$ for all n.

The set R is sequentially compact, so the sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point $a \in R$.

Since{ r_n}converges to r, we know that any subsequence of {r_n } converges to r.

Therefore, the subsequence {r_{n_k}} converges to r.

By Proposition (1.11), we get $\Upsilon(r_{n_k}, a_{n_k}, a_{n_k})$ converges to $\Upsilon(r, a, a)$.

Note that {r_{n_k} } and { a_{n_k} } are subsequences of {r_n}and {a_n }, respectively,

So $\Upsilon(r_{n_k}, a_{n_k}, a_{n_k}) \leq \epsilon$ for all k.

Therefore, we find that

$\Upsilon(r, a, a) \leq \epsilon$.

By the definition of$\mu(r, R, R)$,Then

$\mu(r, R, R) < \epsilon$,so, $r \in : R + \epsilon$

By ourdefinition of $R + \epsilon$. Since r was an arbitrary limit point, then,

$R + \epsilon$,is a closed set since it contains all of its limit points. ⊿

Theorem 2.8

Let $R, S, T \in \mathcal{M}$ and that $\epsilon > 0$. Then $\Gamma(R, S, T) \leq \epsilon$ if and only if $R \subseteq S + \epsilon$, $S \subseteq T + \epsilon$ and$T \subseteq R + \epsilon$.

Proof

By symmetry, it is sufficient. To prove $\rho(S, S, R) \leq \epsilon$, if and only if $S \subseteq R + \epsilon$.

Suppose $S \subseteq R + \epsilon$. By definition of the set $R + \epsilon$, for all $s \in S$,

Then

$\mu(s, s, R) \leq \epsilon$

It follows that$\rho(S, S, , R) \leq \epsilon$.

Now suppose $\rho(S, S, , R) \leq \epsilon$. Then for all $s \in S$,

$\mu(s, s, , R) \leq \epsilon$.

It follows by definition of the set $R + \epsilon$,that $S \subseteq R + \epsilon$.

Similarly, we can prove that $R \subseteq S + \epsilon$, and $T \subseteq R + \epsilon$.

Proposition 2.9:Let $\{R_n\}$ be a Υ-Cauchy sequence in k and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{r_{nk}\}$ is a Υ-Cauchy sequence in $\mathcal{M}$ for which$r_{nk} \in R_{nk}$ for all k, then there exists a Υ-Cauchy sequence $\{y_n\}$ in $\mathcal{M}$ such that$\{y_n\} \in R_n$ for all n, and,$y_{nk} = r_{nk}$, for all k.

Proof

Suppose $\{r_{nk}\}$ is a Υ-Cauchy sequence in $\mathcal{M}$ for which$r_{nk} \in R_{nk}$ for all k.

Define $n_0 = 0$, for each n that satisfy $n_{k-1} < n \leq n_k$. Use Property (iii) to choose $y_n \in R_n$such that $\mu(r_{n_k}, R_n, R_n) = \Upsilon(r_{n_k}, y_n, y_n)$, then using the definitions of ρand μ, we find

$$\Upsilon(r_{n_k}, y_n, y_n) = \mu(r_{n_k}, R_n, R_n) \leq \rho(R_{n_k}, R_n, R_n) \leq \Gamma(R_{n_k}, R_n, R_n).$$

Note that since $r_{n_k} \in R_{n_k}$, then

$$\Upsilon(r_{n_k}, y_{nk}, y_{nk}) = \mu(r_{n_k}, R_{nk}, R_{nk}) = 0$$

It follows that$y_{nk} = r_{n_k}$for all k.Let $\epsilon > 0$, Since $\{r_{nk}\}$ is a ϒ-Cauchy sequence in □, there exists a positive integer K such that

$$\Upsilon(r_{n_k}, r_{nj}, r_{nj}) < \frac{\epsilon}{3} \text{for all } k, j \geq K.$$

Since $\{R_n\}$ is a ϒ-Cauchy sequence in K, by definition there exists a positive integer $N \geq n_k$such that

$$\Gamma(R_n, R_m, R_m) < \frac{\epsilon}{3} \text{ for all } n, m \geq N.$$

Suppose that $n, m \geq N$.

Thenthere existsintegers $j, k \geq K$ such that $n_{k-1} < n \leq n_k$, and $n_{J-1} < m \leq n_j$,

Then we find that

$$\Upsilon(r_n, y_m, y_m) \leq \Upsilon(r_n, r_{n_k}, r_{n_k}) + \Upsilon\left(r_{n_k}, r_{n_j}, r_{n_j}\right) + \Upsilon\left(r_{n_j}, y_m, y_m\right)$$

$$= \mu(r_{n_k}, R_n, R_n) + \Upsilon\left(r_{n_k}, r_{n_j}, r_{n_j}\right) + \Upsilon\left(r_{n_j}, y_m, y_m\right)$$

$$\leq \rho(R_{n_k}, R_n, R_n) + \Upsilon\left(r_{n_k}, r_{n_j}, r_{n_j}\right) + \rho\left(R_{n_j}, R_n, R_n\right)$$

$$\leq \Gamma(R_{n_k}, R_n, R_n) + \Upsilon\left(r_{n_k}, r_{n_j}, r_{n_j}\right) + \Gamma\left(R_{n_j}, R_m, R_m\right)$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$= \epsilon$$

therefore, the set $\{y_n\}$ is a ϒ- Cauchy sequence in $\mathcal{M}$such that $y_n \in R_n$for all n and

$y_{nk} = r_{n_k}$for all n. Since a subset R is compact if and only if it is closed and totally boundedso, we need to prove the

following

Proposition 2.10

Let $\{R_n\}$ be a sequence in $\mathcal{M}$ and let R be the set of all points $r \in \mathcal{M}$ such thatthere exists a sequence $\{r_n\}$ that converges to r and satisfies$r_n \in R_n$for all n. If$\{R_n\}$ is a ϒ- Cauchy sequence, then the set R is closed and nonempty.

Proof: Let us prove that R is nonempty. Since $\{R_n\}$ is a Cauchy sequence, there exists an integer$n_1$such that

$$\Gamma(R_m, R_n, R_n) < \frac{1}{2^1} = \frac{1}{2} \text{for all } m, n \geq n_1.$$

Similarly, there exists an integer$n_2 > n_1$such that

$\Gamma(R_m, R_n, R_n) < \frac{1}{2^2} = \frac{1}{4}$ for all $m, n \geq n_2$.

Continuing this process we have an increasing sequence $\{n_k\}$ such that

$\Gamma(R_m, R_n, R_n) < \frac{1}{2^k} = \frac{1}{2}$ for all $m, n \geq n_k$.

Let x_{n_1}be a fixed point in R_{n_1}. By Property (2) ofTheorem (2.4), we can choose $x_{n_2} \in R_{n_2}$suchthat

$\Upsilon(x_{n_1}, x_{n_2}, x_{n_2}) = \mu(x_{n_1}, R_{n_2}, R_{n_2})$.

Then by definition of μ, ρ, and h, we find

$\Upsilon(x_{n_1}, x_{n_2}, x_{n_2}) = \mu(x_{n_1}, R_{n_2}, R_{n_2})$

$\leq \rho\left(R_{n_1}, R_{n_2}, R_{n_2}\right)$

$\leq \Gamma\left(R_{n_1}, R_{n_2}, R_{n_2}\right) < \frac{1}{2}$.

Similarly we can choose$x_{n_3} \in R_{n_3}$ such that

$\Upsilon(x_{n_2}, x_{n_3}, x_{n_3}) = \mu(x_{n_2}, R_{n_3}, R_{n_3})$

$\leq \rho\left(R_{n_2}, R_{n_3}, R_{n_3}\right)$

$\leq \Gamma\left(R_{n_2}, R_{n_3}, R_{n_3}\right) < \frac{1}{4}$.

Continuing this process we can construct a sequence $\{x_{n_K}\}$ where each $x_{n_K} \in R_{n_K}$ and for all k,

$\Upsilon(x_{n_K}, x_{k+1}, x_{n_{k+1}}) = \mu(x_{n_k}, R_{n_{k+1}}, R_{n_{k+1}})$.

$\leq \rho\left(R_{n_k}, R_{n_{k+1}}, R_{n_{k+1}}\right)$

$\leq \Gamma\left(R_{n_k}, R_{n_{k+1}}, R_{n_{k+}}\right) < \frac{1}{2^k}$.

By previous properties $\{x_{n_k}\}$ is a Cauchy sequence. Therefore, since $\{x_{n_k}\}$ is a Cauchy sequence and $x_{n_k} \in R_{n_k}$ forall k, by the proposition (2.8). There exists a Cauchy sequence $\{y_n\}$ in $\mathcal{M}$ such that$y_n \in R_n$ for all n and $y_{n_k} = x_{n_k}$ for all k.

Since $\mathcal{M}$ is complete, the Cauchy sequence $\{y_n\}$ converges to a point $y \in \mathcal{M}$. Since$y_n \in R_n$for all n, then by definition of the set, $y \in R$. Therefore R is nonempty.

Now we will prove that R is closed. Suppose r is a limit point of R. Then there exists a sequence $r_k \in R\backslash\{r\}$ that converges to r. Since each $r_k \in R$, there exists a sequence $\{y_n\}$such that

$\{y_n\}$ converges to $r_k$and $y_n \in R_n$for each n. Consequently, there exists an integer n_1 such that $x_{n_1} \in R_{n_1}$ and

$\Upsilon(x_{n_1}, r_1, r_1) < 1$.

Similarly, there exists an integer $n_2 > n_1$ and a point $x_{n_2} \in R_{n_2}$ such that

$\Upsilon(x_{n_2}, r_2, r_2) < \frac{1}{2}$.

Continuing this process we can choose an increasing sequence $\{n_k\}$ of integers such that $\Upsilon(x_{n_k}, r_k, r_k) < \frac{1}{k}$ for all k.Then it follows that

$\Upsilon(x_{n_k}, r, r) \leq \Upsilon(x_{n_k}, r_k, r_k) + \Upsilon(r_k, r, r)$.

Note that as we take k to infinity, the distance between $\{x_{n_k}\}$ and a converges to zero, so it follows that $\{x_{n_k}\}$ converge to r. Every convergent sequence is Cauchy, so it follows that

$\{x_{n_k}\}$ is a Cauchy sequence for which $x_{n_k} \in R_{n_k}$ for all k. The proposition (2.9) guarantees that there exists a Cauchy

sequence $\{y_n\}$ in $\mathcal{M}$ such that $y_n \in \{R_n$ for all n and $y_{n_k} = x_{n_k}$.

Therefore $r \in R$, so R is closed.

Definition 2.11[4]

A set $E \subseteq \mathcal{M}$ is called a bounded set if there exists $r \in \mathcal{M}$ and $\epsilon > 0$ such that

$E \subset B_\Upsilon(r, r, \epsilon)$.

Proposition212

Let $\{D_n\}$ be a sequence of totally bounded sets in $\mathcal{M}$, and let R be any subset of $\mathcal{M}$. if for all $\epsilon > 0$,there existsa positive integer ssuch that $R \subseteq D_s + \epsilon$, then R is totally bounded.

Proof

Let $\epsilon > 0$ chose a positive integer s so that $R \subseteq D_s + \frac{\epsilon}{4}$. Since $D_s$is totally bounded, we can choose a finite set $\{r_i: 1 \leq i \leq q\}$, where$r_i \in D_s$such that

$D_s \subseteq \bigcup_{i=1}^{q} B_\Upsilon\left(r_i, \frac{\epsilon}{4}, \frac{\epsilon}{4}\right)$.

By reordering ther_i's, we may assume that $B_\Upsilon\left(r_i, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \cap R \neq \emptyset$ for $1 \leq i \leq p$ and

$B_\Upsilon\left(r_i, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \cap R = \emptyset$for$p < i$.

Then for all $1 \leq i \leq p$, let $y_i \in B_\Upsilon\left(y_i, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \cap R$. We claim that $R \subseteq \bigcup_{i=1}^{q} B_\Upsilon(y_i, \epsilon, \epsilon)$.

Let $a \in R$ then, $a \in D_s + \frac{\epsilon}{4}$. So, Byproposition (2.4)and part (iii),

$\mu(a, D_s, D_s) \leq \frac{\epsilon}{4}$.

Then there exists$r \in D_s$ such that $\gamma(a, r, r) = \mu(a, D_s, D_s)$.

Then we find that

$\gamma(a, r_i, r_i) \leq \gamma(a, r, r) + \gamma(r, r_i, r_i)$

$\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2}$

$= \epsilon$

So, $r \in B_\Upsilon\left(r_i, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right)$ for some $1 \leq i \leq p$. Thus we have

$y_i \in B_\Upsilon\left(x_i, \frac{\epsilon}{2}, \frac{\epsilon}{2}\right) \cap R.$

Such that $\gamma(x_i, y_i, y_i) < \frac{\epsilon}{2}$. It follows that

$\gamma(a, y_i, y_i) \leq \gamma(a, x_i, x_i) + \gamma(x_i, y_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Therefore, since for each $a \in R$ we found y_i for $1 \leq i \leq p$ such that $a \in B_\Upsilon(y_i, \epsilon, \epsilon)$.

Then it follows that

$R \subseteq \bigcup_{i=1}^{P} B_\Upsilon(y_i, \epsilon, \epsilon).$

Thus by Definition, R is totally bounded.

Theorem 2.13

If $(\mathcal{M}, \Upsilon)$ is complet, then $(k(\mathcal{M}), \Gamma)$ is complet.

Proof

To prove $(k(\mathcal{M}), \Gamma)$ complete we must prove every Υ- Cauchy sequence in the space be Υ-convergent to point in the same space. Let $\{A_k\}$be Υ-Cauchy sequence in $\mathcal{M}$, and let A the set of all points $x \in \mathcal{M}$ such that there is a sequence $\{x_n\}$ that converges to x and $x_n \in A_n$ for all n.We must prove that $A \in \mathcal{M}$ and $\{A_k\}$ converges to A. By proposition (2.10) A is closed and nonempty. Let $\epsilon > 0$, since $\{A_n\}$ is Υ-Cauchy then there exists a positive integer L such that $\Gamma(A_n, A_m, A_l) < \epsilon$ for all $m, n, l \geq L$. Then by theorem (2.8),

$A_l \subseteq (A_m \cup A_n) + \epsilon$ for all $l > m, n \geq L$

let$A_k = A_n \cup A_m$. Then $A_l \subseteq A_k + \epsilon$.

Let $a \in A$ to show $a \in A_k + \epsilon$,fixed $k \geq L$.By definition of the set A, here exists asequence $\{x_i\}$ such that

$x_i \in A_i$for all i and$\{x_i\}$ converges to a. By proposition (2.7),

we know $A_k + \epsilon$ is closed.

Since $x_i \in A_k + \epsilon$ for all i then it follows that $a \in A_k + \epsilon$, this shows that $A \subseteq A_k + \epsilon$.

The set A is a totally bounded (by proposition (2.12)). Hence A is complete, since A aclosed subset of complete

GG- mertic space, and A is nonempty complete and totally bounded, then Aiscompact and $A \in \Upsilon$. Let $\epsilon > 0$.

To show that $\{A_k\}$ convergent to $A \in \Upsilon$. There exists apositive integer L such that

$\Upsilon(A_k, A_k, A) < \epsilon$ *for all* $k \geq L$.

We need show that $A \subseteq A_k + \epsilon$, $A_k \subseteq A + \epsilon$ where $k = \max\{m, n\}$. From the first part of proof we know, there exists

Lsuch that for all $k \geq L$

$A \subseteq A_K + \epsilon$ ... (2.4)

To prove $A_k \subseteq A + \epsilon$, let $\epsilon \geq 0$. Since$\{A_k\}$ is a $\Upsilon -$ Cauchy sequence in Υ

There exists a positive integer L such that for all $l, k \geq L$

$\Upsilon(A_k, A_k, A_l) < \frac{\epsilon}{2}$ ... (2.5)

and there exists a strictly increasing sequence $\{k_i\}$ of positive integer such that $k_1 > L$ and

such that

$\Gamma(A_k, A_k, A_l) < \frac{\epsilon}{2^{-i-1}} \; \forall \, k, l \geq k_i$

We can use property (iii) and proposition (2.8) to get the following:

since $A_k \subseteq A_{n1} + \frac{\epsilon}{2}$, there exists $x_{k1} \in A_{k1}$ such that $\gamma(y, x_{k1}, x_{k1}) \leq \frac{\epsilon}{2}$.

Since $A_{k1} \subseteq A_{n2} + \frac{\epsilon}{4}$, there exists $x_{k2} \in A_{k2}$ such that $\gamma(x_{k1}, x_{k2}, x_{k2}) \leq \frac{\epsilon}{4}$.

Since $A_{k2} \subseteq A_{n3} + \frac{\epsilon}{8}$, there exists $x_{k3} \in A_{k3}$ such that $\gamma(x_{k2}, x_{k3}, x_{k3}) \leq \frac{\epsilon}{8}$.

By continuing this process we get a sequence $\{x_{ki}\}$ such that $x_{ki} \in A_{ki,}$ for all i.

And $\Gamma(A_k, A_k, A_l) < \frac{\epsilon}{2^{-i-1}}$. By proposition (2.2) we find $\{x_{ki}\}$ is a Υ-Cauchy sequence,

so by proposition (2.14) the limit of the sequence a is in A.Additionally we find that

$\gamma(y, x_{ki}, x_{ki}) \leq \gamma(y, x_{k1}, x_{k1}) + \gamma(x_{k1}, x_{k2}, x_{k2}) +. \gamma(x_{k2}, x_{k3}, x_{k3}) + \cdots +$

$\gamma(x_{ki-1}, x_{ki}, x_{ki}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} \dots \dots\dots\dots + \frac{\epsilon}{2^i} < \epsilon$

Since $\gamma(y, x_{ki}, x_{ki}) \leq \epsilon$ for all i it follows that $\gamma(y, a, a) \leq \epsilon$ therefore

$y \in A + \epsilon$, thus there exists L such that$A_k \subseteq A + \epsilon$, so it follows that

.$\Gamma(A_k, A_k, A) < \epsilon \; for \; all \; k \geq L$

and hence $\{A_k\}$ converges to $A \in \mathcal{M}$. Therefore $(\mathcal{M}, \Gamma)$ is complete.

Compactness of K($\mathcal{M}$)

Theorem 2.14

If $(\mathcal{M}, \Upsilon)$ is compact then$(K(\mathcal{M}), \Gamma)$is compact.

Proof

By previous result, we know that $K(\mathcal{M})$ is compact. Since we know that a set is compact if and only if it is complete and totally bounded, We must prove that $K(\mathcal{M})$ is totally bounded. Let $\epsilon > 0$ since $\mathcal{M}$ is totally bounded,there exists a finite set $\{ x_i : 1 \leq i \leq n \}$ such that

$\mathcal{M} \subseteq \cup_{i=1}^{n} B_\Gamma (x_i, x_i, \frac{\epsilon}{3})$

and$x_i \in \mathcal{M}$ for each i. Let $\{ C_k : 1 \leq k \leq 2^{p-1} \}$ be collection of all possible non-empty union of the closure of these balls. Since $\mathcal{M}$ is compact, the closure of each ball is the compact set.

Therefore each C_k is a finite union of compact sets and thus compact. So $C_k \in \mathcal{M}$, we want to show that

$K(\mathcal{M}) \subseteq \cup_{i=1}^{2^{P-1}} B_\Gamma (C_k, C_k, \epsilon)$

To do this, let $Z \in K(\mathcal{M})$. Then we want to show that

$Z \in B_\Gamma (C_k, C_k, \epsilon)$ for some $K(\mathcal{M})$.

Choose $S_Z = \{i: Z \cap \overline{B_\Upsilon(x_\imath, x_\imath, \epsilon)} \neq \emptyset \}$, then choose an index j so that

$C_j = \cup \overline{B_\Upsilon(x_\imath, x_\imath, \frac{\epsilon}{3})}$

Since $Z \subseteq C_j$, (by part (ii) and proposition (2.4)) then we know $\rho(Z, C_j, C_j) = 0$

Now let c be an element in $C_j$then there exists some $i \in S_z$ and $z \in Z$ such that $c, z \in \overline{B_\Upsilon(x_\imath, x_\imath, \frac{\epsilon}{3})}$

This implies that$(c, Z, Z) \leq \frac{2}{3} \epsilon$.

Since our choice of c was arbitrary then we find that$(C_j, C_j, Z) < \frac{2}{3} \epsilon$.

Therefore we find that

$\Gamma(Z, C_j, C_j) = \rho(C_j, C_j, Z) < \epsilon$and thus $Z \in B_\Gamma(C_j, C_j, \epsilon)$.

So, $\mathcal{M}$ is totally bounded, therefore we have proved that if $(\mathcal{M}, \Upsilon)$ is compact then $(K(M), \Gamma)$ is compact.

Finally, we give construct a Cuachy sequence of non-empty compact subsets converges to nonempty compact in the following example

Example 2.15: let $(\mathcal{M}, \Upsilon)$ as in example (1.14) and we define the set S as the unit circle in $\mathcal{M}$, i.e,

$$S = \{p = (x, y): \Upsilon(p, 0,0) = 1 \}$$

And for each k, let $S_k = \{(r, t): r = \frac{1}{4} + \frac{1}{4k}\cos(kt), 0 \leq t \leq 2\pi \}$,

Checking the Hausdorff distance between sets, we see that $\Gamma(S_k, S, S) = \frac{1}{4k}$

We can see that $\{S_k\}$ converges to S as k increases, see figures (1-…)

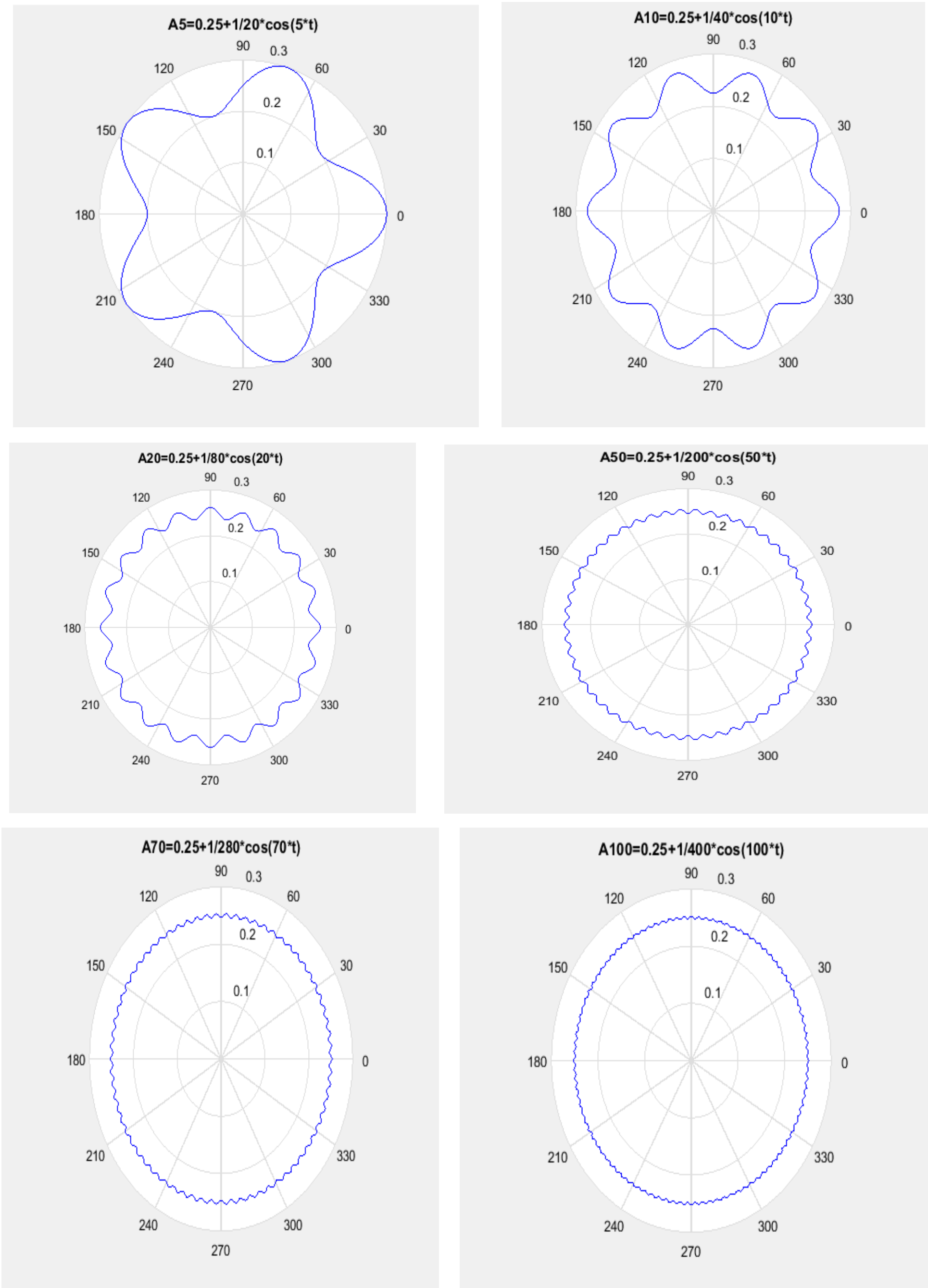

Figure 1

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